

1. (ISSP 5.4)

(a) The density of modes is given by

$$D(\omega)d\omega = 2 \frac{2\pi k dk}{(2\pi/L)^2} = \frac{A}{\pi v_s^2} \omega d\omega$$

where A is the area of the system, v_s is the sound velocity, and the factor of 2 comes from the two different branches (along a high symmetry direction, the two different polarizations, one longitudinal and one transverse). Therefore

$$U = \int_0^{\omega_D} d\omega D(\omega) \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} = \frac{A\hbar}{\pi v_s^2} \int_0^{\omega_D} d\omega \frac{\omega^2}{e^{\hbar\omega/k_B T} - 1} = \frac{A\hbar}{\pi v_s^2} \left(\frac{k_B T}{\hbar}\right)^3 \int_0^{T_D/T} dx \frac{x^2}{e^x - 1}$$

As $T_D/T \rightarrow \infty$, $U = (\text{const})T^3$ and so $C_V \sim T^2$.

(b) Let z be the direction in which the binding is weak, and let the dispersion be

$$\omega^2 = v_1^2(k_x^2 + k_y^2) + v_2^2 k_z^2$$

with $v_1 \gg v_2$. For *extremely* low temperatures, ie

$$k_B T \ll \hbar v_1 \frac{2\pi}{L}$$

only modes with $k_x = k_y = 0$ will be occupied and so the system is effectively one-dimensional. The transition temperature is (for a best case scenario I choose a high v_1 , such as one might find for graphene)

$$T_* \simeq \frac{\hbar}{k_B} \frac{v_1 2\pi}{a} \frac{1}{N^{1/3}} \simeq (1000\text{K}) 10^{-8} \simeq 10^{-5}\text{K}$$

This is a very low temperature indeed.

For higher temperatures, the surface of constant angular frequency in k -space is a prolate ellipsoid, with an z -axis intersection at ω/v_2 , and x - and y -axes intersections at ω/v_1 . Therefore the number of modes with frequency less than ω (assuming all modes have the same dispersion) is

$$N(\omega) = 3 \frac{\frac{4}{3}\pi(\omega/v_1)^2(\omega/v_2)}{(2\pi/L)^3} = \frac{V}{2\pi^2} \frac{\omega^3}{v_1^2 v_2}$$

Note this $N(\omega)$ has nothing to do with the number of atoms N . Then

$$D(\omega) = \frac{dN(\omega)}{d\omega} = \frac{3V}{2\pi^2} \frac{\omega^2}{v_1^2 v_2}$$

Thus $D(\omega) \sim \omega^2$, and so the heat capacity will scale like T^3 .

As the temperature increases above $(\hbar/k_B)(v_2/a)$, (for graphite this is about 10 K), the surface of constant ω becomes a truncated ellipse. In this regime, if we approximate the surface as a cylinder that expands with ω , the density of states scales like $D(\omega) \sim \omega$, and so $C_V \sim T^2$.

Of course, at still higher temperatures $T \gg (\hbar/k_B)(v_2/a)$, the heat capacity will become a constant consistent with the Dulong-Petit law.

2. (ISSP 5.5)

(a) For a single mode the partition function is

$$Z_\omega = \sum_{n=0}^{\infty} e^{-(n+1/2)\hbar\omega/k_B T} = e^{-(1/2)\hbar\omega/k_B T} \frac{1}{1 - e^{-\hbar\omega/k_B T}} = \frac{1}{2 \sinh(\hbar\omega/2k_B T)}$$

where for the last equals sign I multiplied above and below by $e^{(1/2)\hbar\omega/k_B T}$. Then

$$F_\omega = -k_B T \ln Z_\omega = k_B T \ln 2 \sinh(\hbar\omega/2k_B T)$$

(b)

$$F = \frac{1}{2} B \Delta^2 + k_B T \sum_k \ln \left[2 \sinh \left(\frac{\hbar\omega_k(\Delta)}{2k_B T} \right) \right]$$

Minimising with respect to Δ gives

$$0 = B\Delta + k_B T \sum_k \frac{\cosh(\hbar\omega_k(\Delta)/2k_B T)}{\sinh(\hbar\omega_k(\Delta)/2k_B T)} \frac{\hbar}{2k_B T} \frac{\partial \omega_k(\Delta)}{\partial \Delta} = B\Delta + \sum_k \coth(\hbar\omega_k(\Delta)/2k_B T) \frac{\hbar}{2} (-\gamma \omega_k(\Delta))$$

Therefore F is a minimum when

$$B\Delta = \gamma \sum_k \frac{1}{2} \hbar \omega_k(\Delta) \coth \left(\frac{\hbar\omega_k(\Delta)}{2k_B T} \right)$$

The internal energy in a mode with frequency ω is

$$U_\omega = \frac{1}{Z_\omega} \sum_{n=0}^{\infty} (n+1/2) \hbar \omega e^{-(n+1/2)\hbar\omega/k_B T}$$

If we define $\beta = 1/k_B T$, then

$$U_\omega = -\frac{\partial}{\partial \beta} \ln Z_\omega(\beta) = \frac{\partial}{\partial \beta} \ln 2 \sinh(\beta \hbar \omega / 2) = \frac{1}{2} \hbar \omega \coth \left(\frac{\beta \hbar \omega}{2} \right) = \frac{1}{2} \hbar \omega \coth \left(\frac{\hbar \omega}{2k_B T} \right)$$

Therefore the total internal energy is

$$U = \sum_k \frac{1}{2} \hbar \omega_k \coth \left(\frac{\hbar \omega_k}{2k_B T} \right)$$

and so we can write

$$\Delta = \frac{\gamma U(T, \Delta)}{B}$$

(c) In the Debye model with periodic boundary conditions

$$\omega_k = \omega_D \frac{k}{\pi/a} = \omega_D \frac{2n}{N}$$

where n is an integer. Therefore the dependence of ω_k on Δ can only come about from the dependence of ω_D on Δ . Therefore

$$-\gamma = \frac{\partial}{\partial \Delta} \ln \omega_k = \frac{\partial}{\partial \Delta} \ln \omega_D + 0$$

giving the desired result.

Note that γ should be zero for a harmonic lattice. This is clear if we write $\omega_D \simeq v_s/a \simeq \sqrt{C/m}$. Now imagine squeezing the system. Obviously the mass shouldn't change. The spring constant of a harmonic lattice does not depend on the spacing between the points, and so $\gamma = 0$. For an anharmonic lattice, we have (roughly speaking), something like $V = \frac{1}{2}Cx^2 - gx^3 = \frac{1}{2}(C - 2gx)x^2$. This can be thought of as a system with a spring constant that depends on x , and so we would expect $\gamma \neq 0$.

3. (Heat capacity of light) The density of modes for light is given by

$$D(\omega)d\omega = 2 \frac{4\pi k^2 dk}{(2\pi/L)^3} = \frac{V}{\pi^2 c^3} \omega^2 d\omega$$

and so

$$\begin{aligned} U &= \int_0^\infty d\omega D(\omega) \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} = \frac{V\hbar}{\pi^2 c^3} \int_0^\infty d\omega \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} \\ &= \frac{V\hbar}{\pi^2 c^3} \left(\frac{k_B T}{\hbar}\right)^4 \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{V\hbar}{\pi^2 c^3} \left(\frac{k_B T}{\hbar}\right)^4 \frac{\pi^4}{15} \end{aligned}$$

Therefore

$$\frac{U}{V} = \frac{\pi^2}{15} \frac{(k_B T)^4}{(\hbar c)^3}$$

and so the specific heat is

$$\frac{C_V}{V} = \frac{1}{V} \frac{\partial U}{\partial V} = \frac{4\pi^2}{15} \frac{k_B^4 T^3}{(\hbar c)^3}$$

Now the phonon gas has 3 modes at every k point rather than 2, and has a different velocity; otherwise the calculation of the heat capacity for $T \ll T_D$ is the same. Therefore

$$\frac{C_V(\text{photons at } T)}{C_V(\text{phonons at } 1\text{K})} = \frac{2T^3/c^3}{3(1^3)/c_s^3} = \frac{2}{3} \left(\frac{c_s}{c}\right)^3 \left(\frac{T}{1\text{K}}\right)^3$$

For this to be equal to 1 we need

$$\frac{T}{1\text{K}} = \frac{c}{c_s} \left(\frac{3}{2}\right)^{1/3} = \frac{c}{c_s} 1.14$$

Note that $c_s = a\omega_D/\pi = ((10^{28})^{1/3} k_B(100))/(\hbar\pi) \simeq 10^4$; then

$$T \sim (1\text{K}) \frac{3 \times 10^8}{10^4} 1.14 \sim 10^4 \text{K}$$

4. (ISSP 6.1) The density of modes for electrons is given by equation 20, ISSP page 140. (Note: beware equation 21 on page 141! The N in that equation is not the number of electrons in the system, but rather the number of orbitals with energy less than ϵ .)

$$D(E) = CE^{1/2}$$

where C is a constant. We can trade C for the number of particles N through

$$N = \int_0^{E_F} dE D(E) = \frac{2}{3} CE_F^{3/2}$$

and so

$$D(E) = \frac{3N}{2E_F} \left(\frac{E}{E_F} \right)^{1/2}$$

Therefore

$$U = \int_0^{E_F} dE D(E) E = \frac{3N}{2E_F^{3/2}} \frac{E_F^{5/2}}{5/2} = \frac{3}{5} NE_F$$