

1. (ISSP 6.2)

(a)

$$p = - \left( \frac{\partial U}{\partial V} \right)_{S,N}$$

At  $T = 0$  the entropy is a constant as  $V$  is varied. Therefore, since  $E_F = CV^{-2/3}$ ,

$$p = -\frac{3}{5}NC(-)\frac{2}{3}V^{-5/3} = \frac{3}{5}NE_F\frac{2}{3}V^{-1} = \frac{2}{3}\frac{U}{V}$$

(b) From part (a),  $p = C'V^{-5/3}$  where  $C'$  is a constant. Then

$$B = -V \frac{\partial p}{\partial V} = \frac{5}{3}p$$

(c)

$$B = \frac{10}{9} \frac{U_0}{V} = \frac{2}{3} n E_F = \frac{2}{3} (1.4 \times 10^{28})(2.12)(1.6 \times 10^{-19}) = 3.17 \times 10^9 \text{ J/m}^3 = 3.17 \text{ GPa}$$

Wikipedia gives the bulk modulus of potassium as 3.1 GPa (presumably at room temperature).

2. (ISSP 6.4)

(a) The sun is 70% hydrogen and 30% helium; therefore about 85% of its mass comes from protons, giving

$$\text{Number of electrons} = \text{Number of protons} = \frac{.85(2 \times 10^{30})}{1.67 \times 10^{-27}} = 1.0 \times 10^{57}$$

Therefore there will be  $10^{57}$  electrons in the star. The electron density is then

$$n = \frac{10^{57}}{\frac{4}{3}\pi(2 \times 10^7)^3} = \frac{3}{32\pi^2} \times 10^{36} = 1.0 \times 10^{34} \text{ m}^{-3}$$

Then  $k_F = (3\pi^2 n)^{1/3} \simeq (30\pi^2 10^{33})^{1/3} = 6.7 \times 10^{11}$ . Then  $v_F = (\hbar/m)k_F \simeq 7 \times 10^7 \text{ m/s}$ . We're on the border of the relativistic region:  $v/c \simeq .2$ . Therefore we can use the non-relativistic limit but only trust our answer up to  $(v/c)^2 \simeq .01$ .

$$E_F = \frac{1}{2} m v_F^2 = \frac{1}{2} 10^{-30} (50 \times 10^{14}) = 25 \times 10^{-16} = 2.5 \times 10^{-15} \text{ J} \simeq 2 \times 10^4 \text{ eV}$$

(b) Since  $k_F = (3\pi^2 n)^{1/3}$ , then  $E_F = \hbar c (3\pi^2 n)^{1/3} \simeq 3\hbar c n^{1/3}$ .(c) Since  $n \simeq 10^{57}/(10^4)^3 = 10^{45}$  and so  $k \simeq 10^{15}$ . Then  $E_F \simeq 10^{-34+8+15} = 10^{-11} \text{ J}$ , and so  $E_F \simeq 10^{-11+19} = 10^8 \text{ eV}$ .

3. (ISSP 6.6) Given that  $E = E_\omega e^{-i\omega t}$ , we look for solutions that oscillate with frequency  $\omega$ , ie  $v = v_\omega e^{-i\omega t}$ :

$$(m(-i\omega) + m/\tau)v_\omega = -eE_\omega$$

Multiply through by  $n(-e)$  and use  $j_\omega = n(-e)v_\omega$ :

$$j_\omega = \frac{ne^2}{m} \frac{1}{-i\omega + 1/\tau} E_\omega = \sigma(0) \frac{1}{-i\omega\tau + 1} E_\omega$$

Multiplying above and below by  $1 + i\omega\tau$  gives

$$j_\omega = \sigma(0) \frac{1 + i\omega\tau}{1 + (\omega\tau)^2} E_\omega$$

Therefore

$$\sigma(\omega) = \sigma(0) \frac{1 + i\omega\tau}{1 + (\omega\tau)^2}$$

4. (ISSP 6.8)

- (a) From homework 5, problem 4 we know that at 0K each electron has on average

$$\frac{3}{5} E_F = \frac{3}{5} \frac{\hbar^2}{2m} \left( 3\pi^2 \frac{1}{\frac{4}{3}\pi r_0^3} \right)^{2/3} = \frac{3}{5} \left( \frac{9\pi}{4} \right)^{2/3} \frac{\hbar^2}{m} \frac{1}{r_0^2} \frac{a_H}{e^2} \frac{e^2}{2a_H} = 2.21 \frac{a_H^2}{r_0^2} (1Ryd) = \frac{2.21}{r_s^2} (1Ryd)$$

- (b) The relevant energy is given by

$$\int_0^{r_0} dr 4\pi r^2 \rho \frac{e}{r} = \frac{-e}{\frac{4}{3}\pi r_0^3} e 4\pi \int_0^{r_0} dr r^2 \frac{1}{r} = -\frac{3}{2} \frac{e^2}{r_0} = -\frac{3}{r_s} \frac{e^2}{2a_H} = -\frac{3}{r_s} (1Ryd)$$

- (c) This is the same as the energy needed to assemble the sphere of radius  $r_0$  and charge  $-e$ . To add a single shell of charge  $dq$  if the sphere has radius  $r$  costs

$$dW = \frac{dqQ(r)\rho}{r}$$

At constant density  $dq = \rho 4\pi r^2 dr$  and so

$$dW = \frac{\rho 4\pi r^2 dr \rho \frac{4}{3}\pi r^3}{r} = \rho^2 \frac{16\pi^2}{3} r^4 dr$$

Therefore the total energy needed to construct the sphere of radius  $r_0$  is

$$W = \rho^2 \frac{16\pi^2}{3} \frac{r_0^5}{5} = \frac{e^2}{\left(\frac{4}{3}\pi r_0^3\right)^2} \frac{16\pi^2}{3} \frac{r_0^5}{5} = \frac{3}{5} \frac{e^2}{r_0} = \frac{6}{5} \frac{e^2}{2a_H} \frac{a_H}{r_0} = \frac{6}{5} \frac{1}{r_s} (1Ryd)$$

(d) So,

$$U(r_s) = \frac{2.21}{r_s^2} - \frac{1.80}{r_s}$$

For  $U' = 0$  we have

$$\frac{2(2.21)}{r_s^3} = 1.8 \Rightarrow r_s = \frac{2(2.21)}{1.8} = 2.45$$

At the minimum

$$U = 2.21 \frac{1.8^2}{4(2.21)^2} - 1.8 \frac{1.8}{2(2.21)} = -\frac{1}{4} \frac{1.8^2}{2.21} = -0.37Ryd$$

The energy of the separated  $H$  atoms is  $-1Ryd$ , and so we don't expect hydrogen to form a metal.