

1. (ISSP 6.9)

Consider the steady state equations for the velocity, equation (52) page 153 of ISSP. Multiplying across by  $n(-e)$ , gives (using  $\vec{j} = n(-e)\vec{v}$  etc),

$$j_x = \sigma_0 E_x - \omega_c \tau j_y \quad j_y = \sigma_0 E_y + \omega_c \tau j_x \quad j_z = \sigma_0 E_z$$

where  $\sigma_0 = ne^2\tau/m$ . Rewriting:

$$\begin{pmatrix} 1 & \omega_c \tau & 0 \\ -\omega_c \tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \sigma_0 \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Inverting:

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \begin{pmatrix} 1 & -\omega_c \tau & 0 \\ \omega_c \tau & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

In a high magnetic field,

$$\sigma_{yx} = \sigma_0 \frac{\omega_c \tau}{1 + (\omega_c \tau)^2} \simeq \frac{\sigma_0}{\omega_c \tau} = \frac{ne^2 \tau}{m} \frac{mc}{eB\tau} = \frac{ne c}{B}$$

2. (a) The Schrödinger equation gives

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = E(\vec{n}, N) \psi$$

which gives

$$E(\vec{n}, k_z) = \epsilon_{\vec{n}} + \frac{\hbar^2 k_z^2}{2m}$$

where  $k_z = 2\pi N$ . The density of states is  $4dk_z/(2\pi/L_z)$ , where the 4 comes from the 2 spins and 2 directions. Then the density of states per unit length is

$$D_n(E) = \frac{2 dk_z}{\pi dE} = \frac{2}{\pi \hbar v_{\vec{n}}(E)}$$

(b) The total density of states is given by the sum over the modes that have  $\epsilon_{\vec{n}} < E$ . Therefore

$$D(E) = \sum_{\vec{n}} \frac{2}{\pi \hbar v_{\vec{n}}(E)} \theta(E - \epsilon_{\vec{n}})$$

3.

$$I = \sum_{\vec{n}} I_{\vec{n}} = \sum_{\vec{n}} \frac{D_n(E_F)}{2} eV e v_{\vec{n}}(E_F) = \sum_{\vec{n}} \frac{1}{\pi \hbar v_{\vec{n}}(E_F)} eV e v_{\vec{n}}(E_F) \theta(E - \epsilon_{\vec{n}}) = \frac{e^2 V}{\pi \hbar} \sum_{\vec{n}} \theta(E - \epsilon_{\vec{n}}) = \frac{e^2 V}{\pi \hbar} n_{\text{occ}}$$

Since  $\hbar = h/2\pi$ ,

$$I = \frac{2e^2}{h} n_{\text{occ}} V$$

You can see data showing conductance quantization on page 535 of ISSP.

4. (a) The number of particles is given by

$$N = \int_0^{\infty} dE D(E) f(E)$$

where  $f$  is the Fermi-Dirac function  $f(E) = (e^{(E-\mu)/k_B T} + 1)^{-1}$ . In the limit described in the problem, and using the form for  $D(E)$  given by equation (20) on page 140 of ISSP

$$\begin{aligned} N &= V \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} e^{\mu/k_B T} \int_0^{\infty} dE \sqrt{E} e^{-E/k_B T} \\ &= V \frac{1}{2\pi^2} \left( \frac{2mk_B T}{\hbar^2} \right)^{3/2} e^{\mu/k_B T} \int_0^{\infty} d\alpha \sqrt{\alpha} e^{-\alpha} \\ &= V \frac{1}{2\pi^2} \left( \frac{2mk_B T}{\hbar^2} \right)^{3/2} e^{\mu/k_B T} \frac{\sqrt{\pi}}{2} \end{aligned}$$

Then

$$r_s^{-3} = \frac{4\pi n}{3} = \frac{4\pi}{3} \frac{1}{2\pi^2} \frac{\sqrt{\pi}}{2} \left( \frac{2mk_B T}{\hbar^2} \right)^{3/2} e^{\mu/k_B T} = \frac{1}{3\sqrt{\pi}} \left( \frac{2mk_B T}{\hbar^2} \right)^{3/2} e^{\mu/k_B T}$$

and so

$$r_s = 3^{1/3} \pi^{1/6} \left( \frac{2mk_B T}{\hbar^2} \right)^{-1/2} e^{-\mu/3k_B T} = 3^{1/3} \pi^{1/6} \hbar (2mk_B T)^{-1/2} e^{-\mu/3k_B T}$$

- (b) For a particle with energy of order  $k_B T$ , the spread in the momenta is  $\Delta p \simeq \sqrt{2mk_B T}$ . Therefore the uncertainty in position is

$$\Delta x \sim \frac{\hbar}{\Delta p} = \frac{\hbar}{\sqrt{2mk_B T}}$$

So we can think of the gas as a set of wavepackets, each of size  $\Delta x$  and separated by, on average, a distance  $r_s$ . If we think of quantum phenomena as due to the wave nature of electrons, we expect these effects to come into play when the electrons can interfere with each other. This happens for  $\Delta x$  of order  $r_s$ , or smaller. So for  $\Delta x \ll r_s$ , this argument suggests that quantum phenomena might not be important.