1. (ISSP 8.1).

   (a) ISSP, page 210, equation (51):

$$E_d = \frac{13.6}{\epsilon^2}\frac{m_e}{m}eV = \frac{13.6}{18^2}0.015eV = 6.3\times 10^{-4}eV$$

   (b) ISSP, page 210, equation (52):

$$a = \frac{0.53\times\epsilon}{m_e/m}\text{Å} = \frac{(0.53)18}{0.015}\text{Å} = 636\text{Å}$$

   (c) The wavefunctions of the impurity electrons need to overlap if they are to form a band: then the density needs to be at least one electron per $(636\text{Å})^3$, ie

$$n = \frac{1}{(636\text{Å})^3} = 1.6\times 10^7 m^{-3}$$

2. (ISSP 8.2).

   (a) ISSP, page 213, equation (53),

$$n = \sqrt{n_o n_d}e^{-E_g/(2k_BT)}$$

   with

$$n_0 = 2\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} = 2\left(\frac{(0.01)(9.1\times 10^{-31})(1.38\times 10^{-23})4}{2\pi(1.05\times 10^{-34})^2}\right)^{3/2} = 3.9{\times}10^{19}m^{-3} = 3.9{\times}10^{13}cm^{-3}$$

   and so $\sqrt{n_0 n_d} = \sqrt{3.9\times 10^{26}} = 1.97\times 10^{13}cm^{-3}$.
   Also, since $4K$ corresponds to $0.345\times 10^{-3}eV$, we have

$$n = (1.97\times 10^{13}cm^{-3})e^{-1/(2(0.345))} = 4.6\times 10^{12}cm^{-3}$$

   (b) In SI units,

$$R_H = 1/ne = \frac{1}{(4.6\times 10^{18})(1.6\times 10^{-19})} = 1.3m^3/C$$

3. (ISSP 8.3).
   Recall that

$$j_x = \sigma_{xx}E_x + \sigma_{xy}E_y \qquad j_y = \sigma_{yx}E_x + \sigma_{yy}E_y$$

   Since the transverse current must be zero, $E_x = -\sigma_{yy}E_y/\sigma_{yx}$, and so

$$j_x = \left(\sigma_{xy} - \frac{\sigma_{xx}\sigma_{yy}}{\sigma_{yx}}\right)E_y$$

   giving

$$R_H = \frac{E_y}{j_xB} = \frac{1}{B\left(\sigma_{xy} - \frac{\sigma_{xx}\sigma_{yy}}{\sigma_{yx}}\right)} = \frac{\sigma_{yx}}{B(\sigma_{xy}\sigma_{yx} - \sigma_{xx}\sigma_{yy})}$$

Note that σ_{xy} and σ_{yx} are of order B, and so $\sigma_{xy}\sigma_{yx}$ is of order B^2 and can be ignored. Therefore

$$R_H = -\frac{\sigma_{yx}/B}{\sigma_{xx}\sigma_{yy}}$$

In the presence of two carriers, and ignoring terms of order B^2, ISSP page 159 equation 64 gives

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \left[\begin{pmatrix} \sigma_e & -\sigma_e\omega_c^e\tau_e & 0 \\ \sigma_e\omega_c^e\tau_e & \sigma_e & 0 \\ 0 & 0 & \sigma_e \end{pmatrix} + \begin{pmatrix} \sigma_h & \sigma_h\omega_c^h\tau_h & 0 \\ -\sigma_h\omega_c^h\tau_h & \sigma_h & 0 \\ 0 & 0 & \sigma_h \end{pmatrix} \right] \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

The second term comes from the hole carriers. Its derivation is identical to that of electrons, but with τ_e replaced by τ_h, m_e replaced by m_h, and $-e$ replaced by e. Since the cyclotron frequency is always defined to be positive, this gives the above expression. The conductivity matrix is given by the matrix sum enclosed by the square brackets in the previous equation.

Then

$$\begin{aligned} R_H &= -\frac{(\sigma_e\omega_c^e\tau_e - \sigma_h\omega_c^h\tau_h)/B}{(\sigma_e+\sigma_h)^2} = -\frac{ne^2\tau_e^2 e/(m_e^2 c) - pe^2\tau_h^2 e/(m_h^2 c)}{(ne^2\tau_e/m_e + pe^2\tau_h/m_h)^2} \\ &= \frac{\frac{e^3\tau_h^2}{m_h^2 c}\left(p - n\frac{m_h^2 c}{e^3\tau_h^2}\frac{e^3\tau_e^2}{m_e^2 c}\right)}{\left(\frac{e^2\tau_h}{m_h}\right)^2\left(p + \frac{m_h}{e^2\tau_h}\frac{e^2\tau_e}{m_e}n\right)^2} \\ &= \frac{1}{ec}\frac{p - nb^2}{(p+nb)^2} \end{aligned}$$

where

$$b = \frac{e\tau_e/m_e}{e\tau_h/m_h} = \frac{\mu_e}{\mu_h}$$

4. (a)

$$s = \sqrt{\frac{1}{4\pi}} \qquad p_x(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\frac{x}{r} \qquad p_y(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\frac{y}{r} \qquad p_z(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$

Imagine an observer sitting at the point $(1,1,1)$. Looking towards the origin, the observer sees (proceeding clockwise) the x, then z, then y axes, each separated by 120^o.

The action of a rotation on a function is given by $(Rf)(\vec{r}) = f(R^{-1}\vec{r})$. Therefore, a clockwise rotation by $\pi/3$, which takes the x-axis to the z-axis, takes p_z into p_x. It also takes p_y into p_z, and p_x into p_y. Therefore

$$\phi_2 = \frac{1}{2}(s - p_x - p_y + p_z) \to \frac{1}{2}(s - p_y - p_z + p_x) = \phi_3$$

Similarly for ϕ_3 and ϕ_4.

(b) Consider $|\phi_1|^2(\vec{r})$:

$$|\phi_1|^2(\vec{r}) = \frac{1}{4\pi}(1 + 3\hat{n}_1 \cdot \vec{r})^2$$

where $\hat{n}_1 = (1/\sqrt{3})(1,1,1)$. Clearly $|\phi_1|^2(\vec{r})$ is a maximum when $\vec{r}$ points along $\hat{n}_1$, and a minimum when $\vec{r}$ points opposite to $\hat{n}_1$. Identical forms hold for $|\phi_{2,3,4}|^2(\vec{r})$ with $\hat{n}_2 = (1/\sqrt{3})(-1,-1,1)$, $\hat{n}_3 = (1/\sqrt{3})(1,-1,-1)$, and $\hat{n}_4 = (1/\sqrt{3})(-1,1,-1)$.

Therefore the function $|\phi_j|^2(\vec{r})$ has a maximum along $\hat{n}_j$. Since the $\hat{n}_j$ point along the four directions of the tetrahedron, so do the functions $|\phi_j|^2(\vec{r})$.

(c) Since each of the s and p wavefunctions are mutually orthogonal and normalised to 1, then

$$\int d^3r|\phi_1|^2 = \frac{1}{4}\left(\int d^3r|s|^2 + \int d^3r|p_x|^2 + \int d^3r|p_y|^2 + \int d^3r|p_z|^2\right) = \frac{1}{4}(4) = 1$$

Therefore ϕ_1 is normalised to 1; similarly for $\phi_{2,3,4}$.

Note that s, p_x, p_y, and p_z can be constructed from the ϕ_j by inverting the linear transformation given:

$$\begin{pmatrix} s \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

(d) The angle between any two is given by $\theta_{ij} = \cos^{-1}(\hat{n}_i \cdot \hat{n}_j)$. Therefore

$$\theta_{12} = \cos^{-1}((1/3)(-1-1+1)) = \cos^{-1}(-1/3) \simeq 109^o$$

The same result follows for any choice of $i, j = 1,2,3,4$ as long as $i \neq j$.