Physics 240B, Spring 2008
Homework 3 Solutions

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Solutions are in general not the original work of the author.

Problem 1)

The "Ultimate efficiency" of a single-gap solar cell.

a)

The total power is the intensity times the area. Assuming all the power from the blackbody is concentrated on an area of 1 m^2, the incident power is $I_\gamma = \sigma T^4 = 7.36 \times 10^7 W/m^2$. This can also be done using the Planck formula to derive $\sigma = \frac{2\pi^5}{15}\frac{k_B^4}{c^2h^3}$, arriving at the same numerical answer. So the incident power is $P_{in} = 7.36 \times 10^7 W/m^2$.

b)

I_γ computed in (a) is much, much greater than 1000 W/m^2. The single largest contributing factor to this is that the solid angle subtended by the earth of the sun's outgoing power is very, very small. This solid angle is

$$d\Omega = \frac{\pi R^2}{(1A.u.)^2} = \pi \left(\frac{0.696}{149.6}\right)^2 = 6.8 \times 10^{-5}\text{steradians} \tag{1}$$

The fraction of the intensity reaching the earth is $f_\Omega = d\Omega/\pi$, giving a total incident power of $f_\Omega I_\gamma = 1600 W/m^2$.

c)

$$P_{out} = h\nu_g N(\nu_g) A \tag{2}$$

where $E_g = h\nu_g$ is the energy gap, $N(\nu_g)$ is the number of photons incident with energy greater than the bandgap and $A = 1m^2$.

For a blackbody with temperature $T (= 6000K)$, the number of photons with energy greater than the gap is given by the Planck expression

$$N(\nu_g) = \frac{2\pi}{c}\int_{\nu_g}^{\infty} \frac{\nu^2 d\nu}{e^{\beta h\nu} - 1} \tag{3}$$

d)

The efficiency is the ratio of the output power to the incident power.

$$P_{out}/P_{in} = \frac{\frac{2\pi}{c^2} h\nu_g \int_{\nu_g}^{\infty} \frac{\nu^2 d\nu}{e^{\beta h\nu}-1}}{\frac{2\pi}{c^2 h^3}\frac{1}{\beta^4}\frac{\pi^4}{15}} \tag{4}$$

$$\eta(x_g) = \frac{x_g}{\pi^4/15}\int_{x_g}^{\infty}\frac{x^2 dx}{e^x - 1} \tag{5}$$

where $x_g = \beta h\nu_g$. This can be solved numerically/graphically to yield $x_g = 2.17$, or $\eta = 43.8\%$. This gives a bandgap of 1.12 eV, which very nearly the gap of Si. Unfortunately Si is an indirect-gap semiconductor, so it does not form an "ultimately efficient" solar cell.

Problem 2)

The Free Energy associated with causing a superconductor to go to the normal state is:

$$\Delta F = F_N - F_S = \frac{H_c^2 V}{8\pi} \quad \text{(CGS)} \tag{6}$$

$$\Delta S = -\frac{\partial}{\partial T}\Delta F = -V\frac{H_c}{4\pi}\frac{\partial H_c}{\partial T} \tag{7}$$

Since $\frac{\partial H_c}{\partial T} < 0$, $\Delta S = S_N - S_S > 0$ as it should be.

$$\Delta C = T\frac{\partial}{\partial T}\Delta S = -\frac{VT}{4\pi}\left[\left(\frac{\partial H_c}{\partial T}\right)^2 + H_c\frac{\partial^2 H_c}{\partial T^2}\right] \tag{8}$$

By definition, $\Delta F = 0$ at T_c, therefore $H_c = 0$ at T_c, leaving

$$\Delta C(T_c) = -\frac{VT}{4\pi}\left(\frac{\partial H_c}{\partial T}\right)^2_{T_c} \neq 0 \tag{9}$$

Note: At T_c, $\Delta S = 0 \rightarrow$ No latent heat (2nd order transition).

Problem 3)

The London Equation is:

$$\nabla^2 B - \frac{1}{\lambda^2}B = 0 \tag{10}$$

where $^2 = \frac{mc^2}{4\pi ne^2}$. From the cyclindrical symmetry of the problem, $\mathbf{B} = B(\rho)\hat{\mathbf{z}}$.
Therefore,

$$\frac{\partial^2 B}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial B}{\partial \rho} - \frac{1}{\lambda^2}B = 0 \tag{11}$$

This can be re-written with the dimensionless variable $x = \rho/\lambda$ as

$$\frac{\partial^2 B}{\partial x^2} + \frac{\partial B}{\partial x} - B = 0 \tag{12}$$

This is the modified Bessel Equation with index $\nu = 0$ (see Jackson, p.107-8), which has the generic solution

$$B(\rho) = aI_0\left(\frac{\rho}{\lambda}\right) + bK_0\left(\frac{\rho}{\lambda}\right) \tag{13}$$

where $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions. Since $K_0(x) \to \infty$ as $x \to 0$, $b = 0$. At the boundary of hte superconductor ($\rho = R$):

$$B(R) = aI_0\left(\frac{\rho}{\lambda}\right) = H_0 \tag{14}$$

$$a = \frac{H_0}{I_0\left(\frac{R}{\lambda}\right)} \tag{15}$$

So the solutions for the magnetic field are:

$$\mathbf{B} = \frac{H_0}{I_0\left(\frac{R}{\lambda}\right)} I_0\left(\frac{\rho}{\lambda}\right)\hat{\mathbf{z}}, \quad \rho \leq R \tag{16}$$

$$\mathbf{B} = H_0\hat{\mathbf{z}}, \quad \rho \geq R \tag{17}$$

Ampere's equation gives

$$\mathbf{J} = \frac{c}{4\pi}\nabla \times \mathbf{B} = \frac{c}{4\pi}\frac{H_0}{I_0\left(\frac{R}{\lambda}\right)}\nabla \times \left(I_0\left(\frac{\rho}{\lambda}\right)\hat{\mathbf{z}}\right) \tag{18}$$

$$= \frac{c}{4\pi}\frac{H_0}{I_0\left(\frac{R}{\lambda}\right)}\frac{1}{\lambda}I_1\left(\frac{\rho}{\lambda}\right)(-\hat{\phi}) \tag{19}$$

Where $I_1(x)$ is a modified Bessel function of order $\nu = 1$.

$$\mathbf{J} = \frac{c}{4\pi}\frac{H_0}{I_0\left(\frac{R}{\lambda}\right)}\frac{1}{\lambda}I_1\left(\frac{\rho}{\lambda}\right)(-\hat{\phi}), \quad \rho \leq R \tag{20}$$

$$\mathbf{J} = 0, \quad \rho \geq R \tag{21}$$

Problem 4)

For more detail on this problem, see Kittel *Introduction to Solid State Physics* [6th ed., p.340] and Ashcroft and Mermin *Solid State Physics* [p.748].

Start with Ginzburg-Landau theory:

$$|\psi|^2 = n_s, \quad \psi = |\psi|e^{i\theta(\mathbf{r})} \tag{22}$$

and assuming pairs of charge $(-2e)$ to start:

$$\mathbf{j} = (-2e)\psi^\star\mathbf{v}\psi = -\left[\frac{2e^2}{mc}\mathbf{A} + \frac{e\hbar}{m}\nabla\theta\right]|\psi|^2 \tag{23}$$

note that the velocity operator is $\mathbf{v} = \frac{1}{m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)$ in the presence of a field.

Now take a superconducting ring (donut/torus) and a contour C far from its surface.

We have $\oint \mathbf{j} \cdot d\mathbf{l} = 0$ which gives

$$\frac{2e^2}{mc} \oint_C \mathbf{A} \cdot d\mathbf{l} = -\frac{e\hbar}{m} \oint_C \nabla\theta \cdot d\mathbf{l} \tag{24}$$

Now

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_S \mathbf{B} \cdot d\mathbf{S} = \Phi \quad \text{(flux through } C\text{)} \tag{25}$$

and

$$-\oint_C \nabla\theta \cdot d\mathbf{l} = -\Delta\theta = 2\pi n \tag{26}$$

since ψ must be single valued. Thus:

$$|\Phi| = \frac{nhc}{2e} = n\Phi_0 \tag{27}$$

where $\Phi_0 = 2.0679 \times 10^{-7}$ gauss-cm^2.

Kittel, ISSP has a thorough discussion of persistent currents in superconducting rings.