

Physics 240B, Spring 2008

Homework 4 Solutions

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Solutions are in general not the original work of the author.

Problem 1)

a)

Calculate the mean square radius of a Cooper pair:

The Cooper pair wavefunction is given by:

$$\Psi(r) = \sum_k a(k) e^{ik \cdot r} \quad (1)$$

so the mean square radius is:

$$\langle r^2 \rangle = \frac{\int r^2 |\Psi|^2 dr}{\int |\Psi|^2 dr} \quad (2)$$

The denominator is given by:

$$\int |\Psi|^2 dr = \sum_{k,k'} a^\dagger(k') a(k) \int e^{i(k-k') \cdot r} dr = \sum_k |a(k)|^2 \quad (3)$$

For the numerator we use a trick:

$$\nabla_k \Psi(r) = 0 = \nabla_k \left(\sum_{k'} a(k') e^{ik' \cdot r} \right) = e^{ik \cdot r} [\nabla_k a(k) + ira(k)] \quad (4)$$

Upon summing over k we get

$$r \Psi(r) = i \sum_k e^{ik \cdot r} \nabla_k a(k) \quad (5)$$

and therefore

$$\int r^2 |\Psi|^2 dr = \sum_{k,k'} \nabla_{k'} a^\dagger(k') \nabla_k a(k) \int e^{i(k-k') \cdot r} dr = \sum_k |\nabla_k a(k)|^2 \quad (6)$$

and finally

$$\langle r^2 \rangle = \frac{\sum_k |\nabla_k a(k)|^2}{\sum_k |a(k)|^2} \approx \frac{\int_0^\infty d\epsilon N(\epsilon) \left| \frac{\partial a}{\partial \epsilon} \frac{\partial \epsilon}{\partial k} \right|^2}{\int_0^\infty d\epsilon N(\epsilon) |a|^2} \approx \frac{N(0) \left(\frac{\partial \epsilon}{\partial k} \right)^2 \int_0^\infty d\epsilon \left(\frac{\partial a}{\partial \epsilon} \right)^2}{N(0) \int_0^\infty d\epsilon a^2} \quad (7)$$

Using the approximation

$$V_{k,k'} = -\frac{V}{\Omega} \quad \text{for} \quad \epsilon_k, \epsilon_{k'} < \epsilon_D \quad (8)$$

we get

$$a(k) = \frac{C}{E - 2\epsilon_k}, \quad \text{and} \quad \left(\frac{\partial a}{\partial \epsilon}\right)^2 = \frac{4C^2}{(E - 2\epsilon)^4}, \quad a^2 = \frac{C^2}{(E - 2\epsilon)^2} \quad (9)$$

Also, $\left(\frac{\partial \epsilon}{\partial k}\right)^2 = \frac{2\hbar^2 \epsilon_F}{m} = \hbar^2 v_F^2$, and then

$$\langle r^2 \rangle = 4\hbar^2 v_F^2 \frac{\int_0^\infty \frac{d\epsilon}{(E-2\epsilon)^4}}{\int_0^\infty \frac{d\epsilon}{(E-2\epsilon)^2}} = 4\hbar^2 v_F^2 \frac{\frac{1}{6(E-2\epsilon)^3} \Big|_0^\infty}{\frac{1}{2(E-2\epsilon)} \Big|_0^\infty} = \frac{4}{3} \frac{\hbar^2 v_F^2}{E^2} \quad (10)$$

and

$$\sqrt{\langle r^2 \rangle} = \frac{2}{\sqrt{3}} \frac{\hbar v_F}{|E|} = \frac{\hbar v_F}{\sqrt{3}\epsilon_D} \left[e^{\frac{2}{N(0)V}} - 1 \right] \approx \frac{\hbar v_F}{\sqrt{3}\epsilon_D} e^{\frac{2}{N(0)V}} \quad \text{for} \quad N(0)V \ll 1 \quad (11)$$

So

$$\sqrt{\langle r^2 \rangle} \sim \frac{\hbar v_F}{\Delta} \sim \xi_0 = \text{coherence length} \quad (12)$$

b)

Let $\hbar k_{cm}$ denote the center of mass momentum. Then the Schrodinger equation gives:

$$\left[-\frac{\hbar^2}{2m} (\nabla_1^2 + \nabla_2^2) + V \right] \Psi_{pair} = (E + 2\epsilon_F) \Psi_{pair} \quad (13)$$

or

$$[E_{k+k_{cm}/2} + E_{-k+k_{cm}/2}] a_k + \sum_{k'} a_{k'} V_{kk'} = (E + 2\epsilon_F) a_k \quad (14)$$

where E = binding energy of the pair and $E_k = \frac{\hbar^2 k^2}{2m}$. As in class let's define $\epsilon_k = E_k - \epsilon_F$. Then we obtain

$$a_k =$$

Assuming $V_{kk'} = -\frac{V}{\Omega}$ for $0 \leq E \leq \hbar\omega_D$ we obtain the Cooper pair equation

$$1 = -\frac{V}{\Omega} \sum_{k'}$$

but $(\epsilon_{k+k_{cm}/2} - \epsilon_{-k+k_{cm}/2}) = 2\epsilon_k + \frac{\hbar^2 k_{cm}^2}{4m}$, so, to first order in k_{cm} , the denominator is given by $E - 2\epsilon_k$, which is the same as that with $k_{cm} = 0$.

The effect of $k_{cm} \neq 0$ is on the boundary condition when we replace $\sum_k \rightarrow \int dk' \rightarrow \int_{\epsilon_1}^{\epsilon_2} d\epsilon' d\Omega N(0)$.

Before $\epsilon_1 = 0$ and $\epsilon_2 = \hbar\omega_D$. Now we have

$$\epsilon_1 = \frac{\hbar^2}{2m} [k_F^2 + 2k_F \cdot k_{cm}/2 + k_{cm}^2] - E \approx \hbar v_F k_{cm} \cos \theta/2 \quad (17)$$

to first order in k_{cm} . It turns out that $\epsilon_1 = \hbar v_F k_{cm} |\cos \theta|/2$. The upper limit of the integral is such that $k_2 = \sqrt{k_F^2 + k_D^2} - k_{cm} |\cos \theta|/2$ which leads to

$$\epsilon_2 = \hbar\omega_D - \hbar v_F k_{cm} |\cos \theta|/2 \approx \hbar\omega_D \quad (18)$$

since $\hbar\omega_D$ is the dominating term. Keeping the k_{cm} term in the upper limit would contribute only with a term like k_{cm}^2 in the final expression of E . Therefore, we have

$$1 = -\frac{V}{\Omega} 4\pi \int_0^{\pi/2} d\theta \int_{\hbar v_F k_{cm} \cos \theta/2}^{\epsilon_D} \frac{d\epsilon N(\epsilon)}{E - 2\epsilon} \quad (19)$$

$$= -\frac{V}{\Omega} 4\pi N(0) \int_0^{\pi/2} d\theta \left[-\frac{1}{2} \ln(E - 2\epsilon) \right]_{\hbar v_F k_{cm} \cos \theta/2}^{\epsilon_D} \quad (20)$$

$$\approx \frac{1}{2} \frac{N(0)V}{\Omega} \ln \left[1 - \frac{2\hbar\omega_D}{E - \hbar v_F k_{cm}} \right] \quad (21)$$

For the weak coupling limit we have:

$$E_{k_{cm}} = -2\epsilon_D e^{\frac{2\Omega}{N(0)V}} + \hbar k_{cm} v_F \quad (22)$$

c)

If we just consider $k_{cm} = 0$, the single pair model has an energy gap, but if we consider $k_{cm} \neq 0$, then the single pair model has a continuous spectrum (corresponding to various values of k_{cm}) because as k_{cm} increases from zero, the pair binding energy decreases linearly. If $E_0 = 2\epsilon_D e^{\frac{2\Omega}{N(0)V}} = \hbar k_{cm} v_F$ then

$$k_{cm} = \frac{k_B T_c}{\hbar v_F} \sim \frac{k_B T_c}{\epsilon_F} k_F \sim 10^{-4} \text{cm}^{-1} \sim \frac{1}{\xi_0} \quad (23)$$

d)

For a triplet state the spatial wave function $\Psi(R = r - r')$ has to be antisymmetric under exchange of coordinates r and r' . This means $\Psi(R) = \Psi(-R)$ which implies that Ψ has an angular dependence that contains at least the first odd spherical harmonics Y_l^m . So, if we consider the same model as in the singlet case where $V_{kk'} = -\frac{V}{\Omega}$ and has no angular dependence we get (in the development of the Bethe-Goldstone equation):

$$(2\epsilon_k - E)a_k = V \sum_{0 < k' < k_D} a_{k'} \quad (24)$$

$$= V \sum_{k' \text{ in half sphere}} [a_{k'} + a_{-k'}] = 0 \quad (25)$$

So, there is no nontrivial solution for this equation and there is no binding energy. The triplet state can be favored if the potential has a more attractive component with angular dependence $V_{l=1}$. This is the case in He^3 superfluid.

Problem 2)

a)

$$H_{BCS} = H_0 + H_{int} \quad (26)$$

and

$$\Psi = \prod_k [(u_k + v_k b_k^\dagger)] \Phi_0 \quad (27)$$

where

$$H_0 = 2 \sum_k \epsilon_k b_k^\dagger b_k, \quad H_{int} = \sum_{k,k'} V_{kk'} b_{k'}^\dagger b_k^\dagger b_k b_{k'} = n_{k\uparrow} n_{-k\downarrow}, \quad b_k b_k^\dagger = (1 - n_{k\uparrow})(1 - n_{-k\downarrow}) \quad (28)$$

So,

$$\langle \Psi | H_0 | \Psi \rangle = 2 \sum_k \epsilon_k \langle \Phi_0 | \prod_l (u_l + v_l b_l) b_k^\dagger b_k (u_l + v_l b_l^\dagger) | \Phi_0 \rangle \quad (29)$$

For $l \neq k$ we have the factors:

$$(u_l + v_l b_l)(u_l + v_l b_l^\dagger) = u_l^2 + u_l v_l (b_l + b_l^\dagger) + v_l^2 b_l b_l^\dagger \quad (30)$$

which insides the brackets give 1. Therefore

$$\langle \Psi | H_0 | \Psi \rangle = 2 \sum_k \epsilon_k \langle \Phi_0 | (u_k + v_k b_k) b_k^\dagger b_k (u_k + v_k b_k^\dagger) | \Phi_0 \rangle \quad (31)$$

$$= 2 \sum_k \epsilon_k \langle \Phi_0 | u_k^2 b_k^\dagger b_k + u_k v_k (b_k^\dagger b_k b_k^\dagger + b_k b_k^\dagger b_k) + v_k^2 b_k b_k^\dagger b_k b_k^\dagger | \Phi_0 \rangle \quad (32)$$

$$= 2 \sum_k \epsilon_k v_k^2 \quad (33)$$

Now,

$$\langle \Psi | H_{int} | \Psi \rangle = \sum_{k,k'} V_{kk'} \langle \Phi_0 | \prod_l (u_l + v_l b_l) b_{k'}^\dagger b_k (u_l + v_l b_l^\dagger) | \Phi_0 \rangle \quad (34)$$

Again, all factors $l \neq k, k'$ give 1 inside the bracket and we get

$$\langle \Psi | H_{int} | \Psi \rangle = \sum_{k,k'} V_{kk'} \langle \Phi_0 | (u_k + v_k b_k)(u_{k'} + v_{k'} b_{k'}) b_{k'}^\dagger b_k (u_k + v_k b_k^\dagger)(u_{k'} + v_{k'} b_{k'}^\dagger) | \Phi_0 \rangle \quad (35)$$

When k' the matrix element is

$$\langle \dots \rangle = \langle \Phi_0 | (u_k + v_k b_k) b_k (u_k + v_k b_k^\dagger) (u_{k'} + v_{k'} b_{k'}) b_{k'}^\dagger (u_{k'} + v_{k'} b_{k'}^\dagger) | \Phi_0 \rangle \quad (36)$$

$$= \langle \Phi_0 | (u_k^2 b_k + u_k v_k (b_k b_k^\dagger + b_k^2) + v_k^2 b_k^2 b_k^\dagger) (u_{k'}^2 b_{k'}^\dagger + u_{k'} v_{k'} (b_{k'}^{\dagger 2} + b_{k'} b_{k'}^\dagger) + v_{k'}^2 b_{k'} b_{k'}^{\dagger 2}) | \Phi_0 \rangle \quad (37)$$

$$= \langle \Phi_0 | (0 + u_k v_k (1 + 0) + 0) (0 + u_{k'} v_{k'} (0 + 1) + 0) | \Phi_0 \rangle \quad (38)$$

$$= u_k v_k u_{k'} v_{k'} \quad (39)$$

When $k = k'$ the matrix element is:

$$= 4u_k^2 v_k^2 \quad (40)$$

But this includes the self-energy terms which are also present in the normal state so we only count one $u_k^2 v_k^2$, and

$$\langle \Psi | H_{BCS} | \Psi \rangle = 2 \sum_k \epsilon_k v_k^2 + \sum_{k,k'} V_{kk'} u_k v_k u_{k'} v_{k'} \quad (41)$$

b)

Neglecting the chemical potential, let $u_k = \sin \theta_k$, $v_k = \cos \theta_k$. Then

$$\langle H_{BCS} \rangle = 2 \sum_k \epsilon_k \cos^2 \theta_k + \sum_{k,k'} V_{kk'} \sin \theta_k \cos \theta_k \sin \theta_{k'} \cos \theta_{k'} \quad (42)$$

$$= 2 \sum_k \epsilon_k \cos^2 \theta_k + \frac{1}{4} \sum_{k,k'} V_{kk'} \sin 2\theta_k \sin 2\theta_{k'} \quad (43)$$

$$\frac{\partial}{\partial \theta_l} \langle H_{BCS} \rangle = -2\epsilon_l \sin 2\theta_l + \sum_k V_{l,k} \sin 2\theta_k \cos 2\theta_l = 0 \quad (44)$$

$$\epsilon_k \tan 2\theta_k = \frac{1}{2} \sum_{k'} V_{kk'} \sin 2\theta_{k'} = -\Delta_k \quad (45)$$

Defining $E_k = \sqrt{\epsilon_k^2 + \Delta_k^2}$ together with $\tan 2\theta_k = -\frac{\Delta_k}{\epsilon_k}$ and $\sin 2\theta_k = \frac{\Delta_k}{E_k}$, we get the BCS gap equation:

$$\Delta_k = -\frac{1}{2} \sum_{k'} V_{kk'} \frac{\Delta_{k'}}{E_{k'}} \quad (46)$$

Also,

$$u_k^2 = \sin^2 \theta_k = \frac{1}{2}(1 - \cos 2\theta_k) = \frac{1}{2} \left(1 + \frac{\epsilon_k}{E_k} \right) \quad (47)$$

$$v_k^2 = \cos^2 \theta_k = \frac{1}{2}(1 + \cos 2\theta_k) = \frac{1}{2} \left(1 - \frac{\epsilon_k}{E_k} \right) \quad (48)$$

$$u_k v_k = \sin \theta_k \cos \theta_k = \frac{1}{2} \sin 2\theta_k = \frac{\Delta_k}{2E_k} \quad (49)$$

Assuming that the potential is constant and attractive inside a with ϵ_D of the Fermi energy. We want a solution such that $\Delta_K = \Delta$ for $|\epsilon_k| < \hbar\omega_D$.

Then, the gap equation becomes,

$$\frac{2}{V} = \sum_k \frac{1}{\sqrt{\epsilon_k^2 + \Delta^2}} \approx \int_{-\hbar\omega}^{\hbar\omega} \frac{N(\epsilon) d\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} \approx N(0) \int_{-\hbar\omega}^{\hbar\omega} \frac{d\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} \quad (50)$$

$$= N(0) \ln \left[\frac{\sqrt{(\hbar\omega)^2 + \Delta^2} + \hbar\omega}{\sqrt{(\hbar\omega)^2 + \Delta^2} - \hbar\omega} \right] \quad (51)$$

$$\rightarrow \Delta = \frac{\hbar\omega}{\sinh \frac{1}{N(0)V}} \approx 2\hbar\omega e^{-\frac{1}{N(0)V}} \quad (\text{with } N(0)V \ll 1) \quad (52)$$

For the energy difference between the normal and superconductor states we get

$$W_N - W_S = \langle H_{BCS} \rangle|_{\Delta=0} - \langle H_{BCS} \rangle|_{\Delta \neq 0} \quad (53)$$

$$\begin{aligned} &= 2 \sum_k \epsilon_k v_k^2 |_{\Delta=0} - 2 \sum_k \epsilon_k v_k^2 |_{\Delta \neq 0} - \sum_{k,k'} V_{kk'} u_k v_k u_{k'} v_{k'} |_{\Delta \neq 0} \quad (54) \\ &= \sum_k \epsilon_k \left(1 - \frac{\epsilon_k}{|\epsilon_k|} \right) - \sum_k \epsilon_k \left(1 - \frac{\epsilon_k}{E_k} \right) + \frac{V\Delta^2}{4} \sum_{k,k'} \frac{1}{E_k E_{k'}} \quad (55) \end{aligned}$$

$$= \sum_k \left(-|\epsilon_k| + \frac{\epsilon_k^2}{E_k} \right) + \frac{V\Delta^2}{4} \sum_{k,k'} \frac{1}{E_k E_{k'}} \quad (56)$$

We saw that $\sum_k \frac{1}{E_k} = \frac{2}{V}$ so,

$$W_N - W_S = \sum_k \left(-|\epsilon_k| + \frac{\epsilon_k^2}{E_k} + \frac{\Delta^2}{2E_k} \right) \quad (57)$$

$$= N(0) \int_{-\hbar\omega}^{\hbar\omega} d\epsilon \left(-|\epsilon| + \frac{\epsilon^2}{E} + \frac{\Delta^2}{2E} \right) \quad (58)$$

$$\approx 2N(0) \int_0^{\infty} d\epsilon (-\epsilon +$$

$$= 2N(0) \left[-\frac{\epsilon^2}{2} + \frac{\epsilon\sqrt{\epsilon^2 + \Delta^2}}{2} \right]_0^{\infty} \quad (60)$$

$$= \frac{N(0)\Delta^2}{2} \quad (61)$$

Problem 3)

For the 3 square well model, find the transition temperature and show that when the exciton Kernel is eliminated, the solution reduces to the two square well model.

From the gap equation:

$$D(\epsilon) = - \int_{\hbar\omega_c}^{\hbar\omega_c} \frac{D(\epsilon')}{\epsilon'} K(\epsilon, \epsilon') \tanh \left[\frac{\epsilon'}{2K_B T_c} \right] d\epsilon' \quad (62)$$

and using

$$K_1 = (K_c - |K_p| - |K_e|)/2 \quad (63)$$

$$K_2 = (K_c - |K_e|)/2 \quad (64)$$

$$K_3 = K_c/2 \quad (65)$$

together with the definitions, from the gap integrals

$$Z_1 = \ln \left(\frac{1.14\epsilon_1}{K_B T_c} \right) \quad (66)$$

$$Z_2 = \ln \left(\frac{\epsilon_2}{\epsilon_1} \right) \quad (67)$$

$$Z_3 = \ln \left(\frac{\epsilon_3}{\epsilon_2} \right) \quad (68)$$

we obtain

$$-D_1 = D_1 K_1 Z_1 + D_2 K_2 Z_2 + D_3 K_3 Z_3 \quad (69)$$

$$-D_2 = D_1 K_2 Z_1 + D_2 K_2 Z_2 + D_3 K_3 Z_3 \quad (70)$$

$$-D_3 = D_1 K_3 Z_1 + D_2 K_3 Z_2 + D_3 K_3 Z_3 \quad (71)$$

This system of equations has no solution unless the determinant vanishes. i.e.:

$$[(K_1 - K_2)Z_1 + 1][(K_2 Z_2 + 1)(K_3 Z_3 + 1) - K_3^2 Z_2 Z_3] + [K_2 Z_1 (K_3 Z_3 + 1) - K_3^2 Z_3 Z_1] = 0 \quad (72)$$

which gives

$$-Z_1 = \left[-K_p - \frac{K_3^2 Z_3 - K_2 (K_2 Z_3 + 1)}{(K_2 Z_2 + 1)(K_3 Z_3 + 1) - K_3^2 Z_2 Z_3} \right]^{-1} \quad (73)$$

If $K_e = 0$ then $K_2 = K_3 = K_c$ and we obtain

$$-Z_1 = \left[-K_p - \frac{K_c}{K_c \ln(\epsilon_3/\epsilon_1) + 1} \right]^{-1} \quad (74)$$

i.e.

$$Z_1 = \ln \left(\frac{1.14\epsilon_1}{K_B T_c} \right) = \frac{1}{K_p - K_c^*}, \quad \text{with} \quad K_c^* = \frac{K_c}{1 + K_c \ln(\epsilon_3/\epsilon_1)} \quad (75)$$

which is the two square-well model solution.

Problem 4)

a)

The operator for the electron-phonon interaction is

$$V_{eph} = \sum_{k,q} M_q (a_q + a_{-q}^\dagger) c_{k+q}^\dagger c_k \quad (76)$$

The electron-electron interaction by phonon exchange is V_{ee}^p , and from lecture:

$$\langle i | V_{ee}^p | f \rangle = \frac{1}{2} \sum_j \langle i | V_{eph} | j \rangle \langle j | V_{eph} | f \rangle \left[\frac{1}{E_f - E_j} + \frac{1}{E_i - E_j} \right] \quad (77)$$

where the initial state is give by

$$|i\rangle = |k, -k, n_q, n_{-q}\rangle \quad (78)$$

and the final state:

$$|f\rangle = |k + q, -k - q, n_q, n_{-q}\rangle \quad (79)$$

and $|j\rangle$ is any possible intermediate state (here n_q is the number of phonons in mode $q \neq 0$ since $T \neq 0$).

The possible intermediate states, $|j\rangle$, are:

$$|j_1\rangle = |k + q, -k, n_q, n_{-q} + 1\rangle \quad (80)$$

$$|j_2\rangle = |k, -k - q, n_q + 1, n_{-q}\rangle \quad (81)$$

$$|j_3\rangle = |k + q, -k, n_q - 1, n_{-q}\rangle \quad (82)$$

$$|j_4\rangle = |k, -k - q, n_q, n_{-q} - 1\rangle \quad (83)$$

where the first two are still possible as $T \rightarrow 0$, since the mediating phonon is emitted first, while the second two, where the absorption is the first process, no longer participate at $T = 0$.

Now, assuming $\epsilon(k) = \epsilon(-k)$ and $\omega_q = \text{omega}_{-q}$, we have the energy differences for each intermediate state:

$$|j_1\rangle : E_f - E_j = \epsilon_{k+q} - \epsilon_k - \hbar\omega_q; \quad E_i - E_j = -(\epsilon_{k+q} - \epsilon_k) - \hbar\omega_q \quad (84)$$

$$|j_2\rangle : E_f - E_j = \epsilon_{k+q} - \epsilon_k - \hbar\omega_q; \quad E_i - E_j = -(\epsilon_{k+q} - \epsilon_k) - \hbar\omega_q \quad (85)$$

$$|j_3\rangle : E_f - E_j = \epsilon_{k+q} - \epsilon_k + \hbar\omega_q; \quad E_i - E_j = -(\epsilon_{k+q} - \epsilon_k) + \hbar\omega_q \quad (86)$$

$$|j_4\rangle : E_f - E_j = \epsilon_{k+q} - \epsilon_k + \hbar\omega_q; \quad E_i - E_j = -(\epsilon_{k+q} - \epsilon_k) + \hbar\omega_q \quad (87)$$

The Matrix Elements (Doing $|j_1\rangle$ as an example, the others follow):

$$V_{eph}|f\rangle = \sum_{k', q'} M_{q'} (a_{q'} + a_{-q'}^\dagger) c_{k'+q'}^\dagger c_{k'} |k + q, -k - q, n_q, n_{-q}\rangle \quad (88)$$

For $|j_1\rangle$ we need parts in the sum that destroy the electron at $-k - q$, create one at $-k$, and create a phonon at $-q \rightarrow k' = -k - q, q' = q$

Thus, the only term of interest in $V_{eph}|f\rangle$ is

$$M_q (a_q + a_{-q}^\dagger) c_{-k}^{-k-q}|f\rangle \quad (89)$$

Only the a_{-q}^\dagger is of interest, since the a_q will generate a state orthogonal to $|j_1\rangle$.

$$a_{-q}^\dagger |-, -, n_q, n_{-q}\rangle = \sqrt{n_{-q} + 1} |-, -, n_q, n_{-q} + 1\rangle \rightarrow \langle j_1 | V_{eph} | f \rangle = \sqrt{n_{-q} + 1} M_q \quad (90)$$

After some algebra:

$$\langle i | V_{eph} | j_1 \rangle = M_{-q} \sqrt{n_{-q} + 1} \quad \langle j_1 | V_{eph} | f \rangle = M_q \sqrt{n_{-q} + 1} \quad (91)$$

$$\langle i | V_{eph} | j_2 \rangle = M_q \sqrt{n_q + 1} \quad \langle j_2 | V_{eph} | f \rangle = M_{-q} \sqrt{n_q + 1} \quad (92)$$

$$\langle i | V_{eph} | j_3 \rangle = M_{-q} \sqrt{n_q} \quad \langle j_3 | V_{eph} | f \rangle = M_q \sqrt{n_q} \quad (93)$$

$$\langle i | V_{eph} | j_4 \rangle = M_q \sqrt{n_{-q}} \quad \langle j_4 | V_{eph} | f \rangle = M_{-q} \sqrt{n_{-q}} \quad (94)$$

So, we finally have

$$\langle i|V_{ee}^p|f\rangle = \frac{1}{2}M_qM_{-q}(n_{-q} + 1 + n_q + 1) \left(\frac{1}{\epsilon_{k+q} - \epsilon_k - \hbar\omega_q} + \frac{1}{-(\epsilon_{k+q} - \epsilon_k) - \hbar\omega_q} \right) \quad (95)$$

$$+ \frac{1}{2}M_qM_{-q}(n_q + n_{-q}) \left(\frac{1}{\epsilon_{k+q} - \epsilon_k + \hbar\omega_q} + \frac{1}{-(\epsilon_{k+q} - \epsilon_k) + \hbar\omega_q} \right) \quad (96)$$

Remembering that $M_q = M_{-q}$, so $M_qM_{-q} = |M_q|^2$, the above term becomes

$$\langle i|V_{ee}^p|j\rangle = |M_q|^2 \frac{2\hbar\omega_q}{(\epsilon_{k+q} - \epsilon_k)^2 - (\hbar\omega_q)^2} \quad (97)$$

which is independent of temperature.

b)

The Electron-phonon self-energy: Using 2nd order perturbation theory, there are two intermediate states giving non-zero matrix elements for V_{eph} :

$$|1\rangle = |k + q, \{\dots, n_{-q} + 1, \dots\}\rangle \quad (98)$$

$$|2\rangle = |k + q, \{\dots, n_{-q} - 1, \dots\}\rangle \quad (99)$$

Meaning one diagram emits and absorbs a phonon of wavevector q , while the other first absorbs, then emits the phonon.

$$E_\psi - E_1 = \epsilon_k - \epsilon_{k+q} - \hbar\omega_q \quad (100)$$

$$E_\psi - E_2 = \epsilon_k - \epsilon_{k+q} + \hbar\omega_q \quad (101)$$

The Matrix elements are given by:

$$\langle 1|V_{eph}|\psi\rangle = M_q(1 - N_{k+q})\sqrt{n_{-q} + 1} \quad (102)$$

$$\langle 2|V_{eph}|\psi\rangle = M_q(1 - N_{k+q})\sqrt{n_q} \quad (103)$$

Where N_{k+q} is an electron occupation number, so $(1 - N_{k+q})^2 = (1 - N_{k+q})$.

$$\Delta E_{eph} = \sum_q (1 - N_{k+q})|M_q|^2 \left[\frac{n_{-q} + 1}{\epsilon_k - \epsilon_{k+q} - \hbar\omega_q} + \frac{n_q}{\epsilon_k - \epsilon_{k+q} + \hbar\omega_q} \right] \quad (104)$$

Using $\langle n_q \rangle = \langle n_{-q} \rangle$ (thermal averages), this simplifies to:

$$\Delta E_{eph} = \sum_q (1 - N_{k+q})|M_q|^2 \frac{\hbar\omega_q - (2\langle n_q \rangle + 1)(\epsilon_{k+q} - \epsilon_k)}{(\epsilon_{k+q} - \epsilon_k)^2 - (\hbar\omega_q)^2} \quad (105)$$

Where $\langle n_q \rangle$ is given by a boson distribution with the chemical potential set to 0. Note that the temperature dependence remains in the final answer.

c)

Find $N(0)V^c$ and $N(0)V^{BP}$ for typical metals (actual numerical estimates).

$$N(0) \sim \frac{3}{2} \frac{n}{\epsilon_F} = \frac{3}{2} n \left(\frac{2m}{\hbar^2 (3\pi^2 n)^{2/3}} \right) = \frac{mk_F}{\pi^2 \hbar^2} \quad (106)$$

$$V^c \sim \frac{4\pi e^2}{k_s^2 + q^2} \quad (107)$$

where k_s is the Fermi-Thomas screening wave-vector. Choose $q \sim q_{Debye} \sim \pi/\text{lattice constant}$, since $k_s^2 \sim q^2$ (roughly), choose $k_s^2 + q^2 \sim 10^6 [\text{cm}^{-2}]$. $k_F \sim 10^8$ for a typical metal so

$$N(0)V^c \sim 1 \quad (108)$$

It should actually be closer to 0.1, but this is just a crude estimate.

Problem 5)

The operators are defined as

$$b_k^\dagger = c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger b_k = c_{k\uparrow} c_{-k\downarrow} \quad (109)$$

where the

$$[c_s, c_{s'}]_+ = [c_s^\dagger, c_{s'}^\dagger]_+ = 0 \rightarrow c_s^2 = c_s^{\dagger 2} = 0 \quad (110)$$

$$[c_s, c_{s'}^\dagger]_+ = \delta_{ss'} \quad \text{and} \quad c_s^\dagger c_s = n_s \quad (111)$$

Therefore

$$[b_k, b_{k'}^\dagger]_- = c_{-k\downarrow} c_{k\uparrow} c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger - c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger c_{-k\downarrow} c_{k\uparrow} \quad (112)$$

$$= c_{k\uparrow} c_{k'\uparrow}^\dagger c_{-k\downarrow} c_{-k'\downarrow}^\dagger - c_{k'\uparrow}^\dagger c_{k\uparrow} c_{-k\downarrow} c_{-k'\downarrow}^\dagger \quad (113)$$

$$(\delta_{kk'} - c_{k'\uparrow}^\dagger c_{k\uparrow}) (\delta_{kk'} - c_{-k'\downarrow}^\dagger c_{-k\downarrow} - c_{k'\uparrow}^\dagger c_{k\uparrow} c_{-k'\downarrow}^\dagger c_{-k\downarrow}) \quad (114)$$

$$= (1 - n_{k\uparrow} - n_{-k\downarrow}) \delta_{kk'} \quad (115)$$

$$b_k^2 = c_{-k\downarrow} c_{k\uparrow} c_{-k\downarrow} c_{k\uparrow} = -c_{-k\downarrow}^2 c_{k\uparrow}^2 = 0 \quad (116)$$

$$b_k^{\dagger 2} = c_{-k\downarrow}^\dagger c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{k\uparrow}^\dagger = -c_{-k\downarrow}^{\dagger 2} c_{k\uparrow}^{\dagger 2} = 0 \quad (117)$$

From the commutator:

$$[b_k, b_{k'}]_- = c_{-k\downarrow} c_{k\uparrow} c_{-k'\downarrow} c_{k'\uparrow} - c_{-k'\downarrow} c_{k'\uparrow} c_{-k\downarrow} c_{k\uparrow} \quad (118)$$

$$= c_{-k'\downarrow} c_{k'\uparrow} c_{-k\downarrow} c_{k\uparrow} - c_{-k'\downarrow} c_{k'\uparrow} c_{-k\downarrow} c_{k\uparrow} \quad (119)$$

$$= 0 \quad (120)$$

we get

$$[b_k, b_{k'}]_+ = 2b_k b_{k'} (1 - \delta_{kk'}) \quad (121)$$

and from $[b_k, b_{k'}^\dagger]_-$ above we get

$$[b_k, b_{k'}^\dagger]_+ = (1 - n_{k\uparrow} - n_{-k\downarrow}) \delta_{kk'} + 2b_{k'}^\dagger b_k \quad (122)$$